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3D printing and laser cutting of architectural heritage for use in mathematics education

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Abstract

The recent introduction of new technologies such as 3D printing or laser cutting has made it possible to support the visualization and manipulation of mathematical objects with easy-to-produce parts. These parts, as opposed to the handcrafted ones, are manufactured from a digital file containing the design of the part.

In this article we focus on mathematical objects derived from architectural and art elements. They are of special interest, both for their mathematical design and for their link to art, to history, and to the emotions produced by their contemplation in the monumental heritage. This, together with the engineering of their design and the technology of their production, makes them a real STEAM material.

We will expose the manufacture of some elements such as muqarnas ceilings, cross ribbed vaults, iconic renaissance elements, islamic mosaics and some more. We will detail the geometric characteristics of these elements, their design process to obtain a 3D or 2D model and the means of fabrication to obtain an operable piece. Finally we will discuss some examples and indications of their use in didactic workshops for mathematics education.

1. Introduction

The historical architectural heritage contains many examples of objects with geometric nature and great visual appeal. These elements have been used to analyze the significance of monuments and also as a resource in mathematics education [8, 9]. In the latter aspect, the historical values, the cultural and emotional connection that they represent and their geometric elaboration over centuries make them a great resource to organize mathematical visits, workshops and collaborative tasks for students.

However, some of these elements are difficult to understand at a glance, even using images and drawing mathematical layers over them (for example using GeoGebra [16]). However, the display of a physical 3D model and its manipulation takes the student to another world of spatial recognition and improves attention, understanding of the object and increases the interest in it.

Fortunately, in recent years, the rising technologies of **3D printing** and **laser cutting,** have emerged helping us to materialize this kind of models. Both technologies are very suitable for reproducing architectural objects, whose main design rules are of geometrical type (curves, tessellations, Euclidean geometry in 2D or 3D, and so on), which may be complex to understand. Moreover, many models can be decomposed in different pieces forming a kind of puzzle, which once assembled emulates the architectural object and shows all its parts and their relationship. To sum up, they are great for spatial understanding and as visual and conceptual stimulation.

In this article we will focus on some intricate and complex architectural heritage objects. We describe how to model and produce them, in order to use them as an aiding tool to explain mathematical concepts that are applied in architecture and arts.

Particularly, we will focus on muqarnas vaults and ribbed domes, typical of Islamic architecture, and some other renaissance and baroque iconic objects. We will also show how to laser cut some of the best known and geometrically pleasing objects, the mosaics of Andalusian decoration. These wood tiles, once painted and mounted, are an excellent tool to teach about math concepts such as reflections, rotations and translations. Even more, using those tiles we can explain the different configurations of the “wallpaper groups”. Last, but not least, we also show how we have modeled and produced some tools to be used in math education in order to manipulate concepts like proportions and the configuration parameters from horseshoe arches.

2. Muqarnas vaults

Muqarnas vaults are some of the most complex 3D geometrical objects in Andalusian architecture. They are produced by aggregation of simpler structures [4, 11, 12]. Muqarnas are marvelous pieces of architectural decoration, which produce amazement and recreate our visual capacity in their geometric interlocking (Fig. 1).

Figure 1. Muqarnas vault. *Dos Hermanas* hall. La Alhambra (Granada).

To produce a realistic model of a muqarnas vault we part from 4 basic prisms with section sizes based on √2 proportions that interlock (Fig. 2). These prisms are called *conza, medio cuadrado* (half square)*, dumbaque* and *jaira* (Fig. 3). In addition to the basic prisms some other singular prisms such as *estrella* (star), *almendrilla* or other special ones are added. The star shaped prism usually forms the central piece or center of local radial symmetries in the vault.

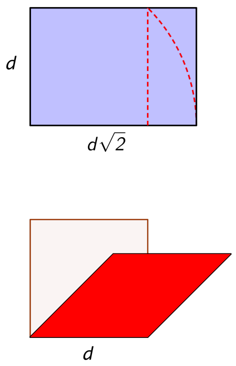


Figure 2. Muqarnas prisms plan.

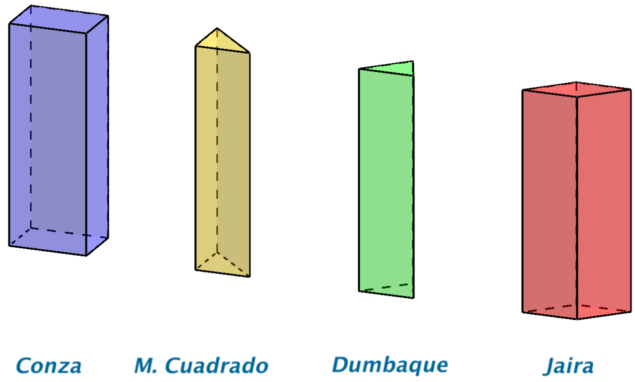
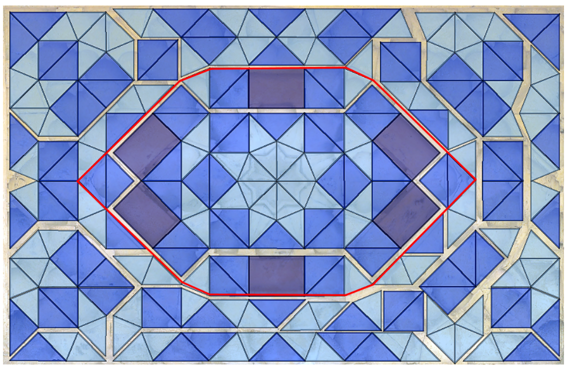


Figure 3. Muqarnas prisms.

All together they cover a tesselation of a rectangular or polygonal area, like those of a vault plan (see Fig. 4). The amount of possible designs to produce this covering of the plane using the 4 prisms is very big. However, all designs must reflect some basic symmetry properties, such as the symmetry of perpendicular central axes in the case of a rectangular plan or radial symmetry in the case of a (regular) polygonal plan. The production of the particular designs we find in monuments is a combination of those symmetry properties and the creativity of the craftsmen.



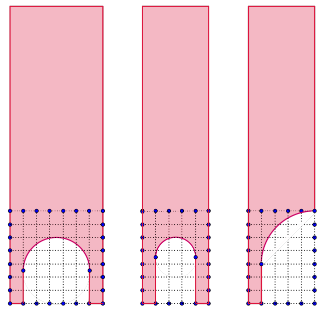


Figure 4 (left). Muqarnas vault plan. Puerta del Lagarto.
Figure 5 (right). Convex cutting of a *conza*.

The basic prisms are cut to produce rounded concave shapes on them. The cuts are precisely defined by ancient treaties, and are based on divisions of 7 and 5 (the quotient of 7/5 closely approximates √2). These cuts leave elongated supports whose size is 1/7 the width of the base piece (if they are on the longest side) or 1/5 (if they are on the shortest). These pins are called *patillas* and they form clusters when the pieces join together. In Fig. 5 we show these cuts for a *conza* piece.

This cutting process makes shapes more pleasing to the eye. The muqarnas are arranged starting from a central piece (usually a star) and adding new pieces around the previous ones. The new layers are arranged at lower levels, which causes them to hang down the walls. This arrangement together with the concave cuts finally give the appearance of a dome-like structure with stalactites (Figs. 6 and 7).

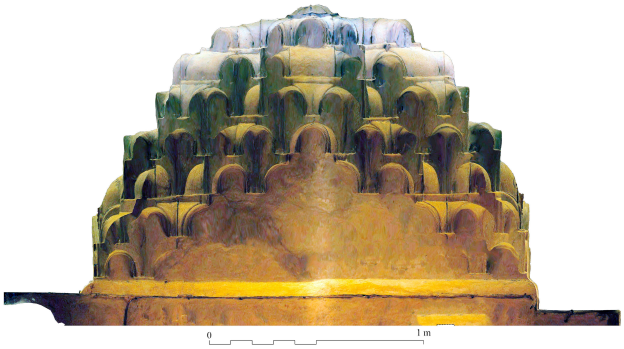


Figure 6 (left). Muqarnas vault. Puerta del Lagarto. Picture from bottom.
Figure 7 (right). *The Lagarto vault* reconstruction with photogrammetric techniques shows the resulting dome-like structure.

Particularly, we have modeled and 3D printed the central part of the oldest and best preserved muqarnas ceilings in Andalusia: the one above the *Puerta del Lagarto* (XII A.D.), in the Cathedral of Seville. This central part comprises 8 different models (Fig. 8) (65 total pieces) that are mounted over a 3 layer foundation (Figs. 9 and 10). The original muqarnas ceiling comprises a total 235 pieces and 7 different layers.

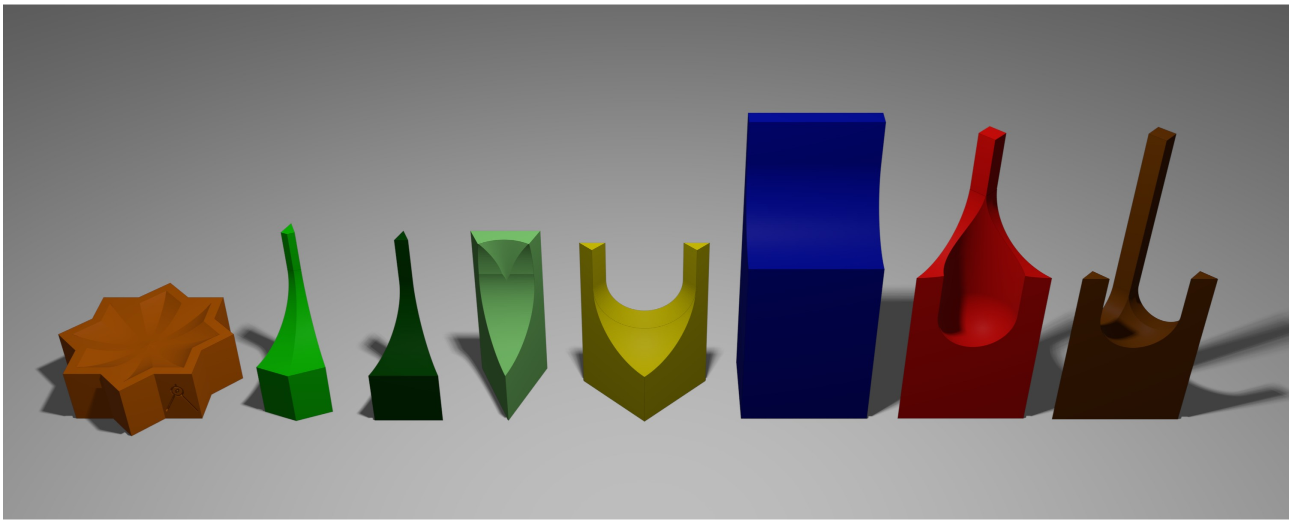
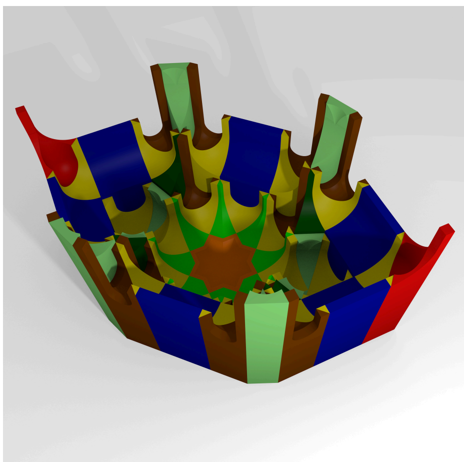


Figure 8. Set of 8 pieces modeled for the central part of the Muqarnas vault at *Puerta del Lagarto*.



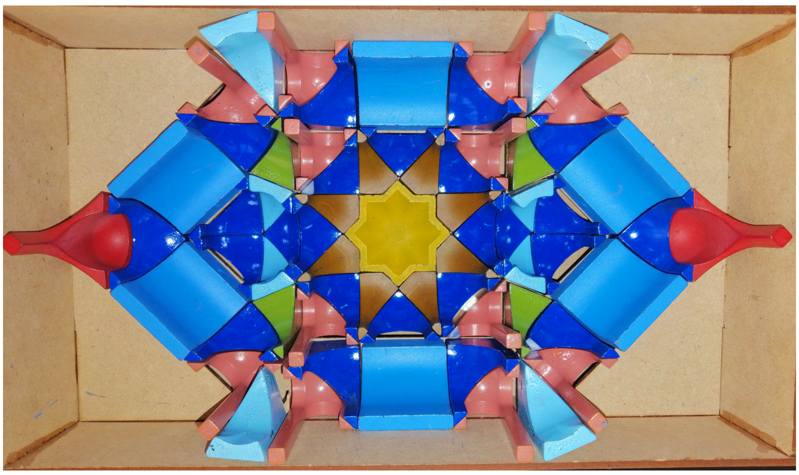


Figure 9 (left). Rendered model of the central part of the Muqarnas vault at *Puerta del Lagarto*.
Figure 10 (right). The 3D printed and mounted central part of the Muqarnas vault.

3. Ribbed domes

A ribbed dome is a dome composed of arches that cross each other, forming the support of the roof. In the Great Mosque of *Córdoba* we can find 3 pioneering ribbed domes whose main purpose is to sustain a central smaller dome (*cupulín*) and allow the entrance of light from the sides [2, 3]. This is an innovative solution in architecture, which until that moment had been solved with simpler types of domes.

All domes have 8 ribs, but differ in the way they cross from one to another. Depending on the placement and interlocking of the arches they provide a greater or lesser space for the *cupulín.* The first to be built, that of the Villaviciosa Chapel, minimizes the number of arch crosses, with only 4 of a lattice + diamond type. The dome of the Maqsura, the most majestic and beautiful, has 8 crosses with 8 equal arches. It forms a stellated polygon of type [8/2] (Schläfli symbol for stellate polygons, where *[p/q]* means that it is built on a regular polygon with *p* sides, joining the vertices separated by a distance *q*), which maximizes the size of the *cupulín* it supports above its arches. The eastern and western lateral domes of the Maqsura, have the maximum number of crosses (12) adopting in their projection the form of a stellated polygon of type [8/3]. In this case the extension of the *cupulín* is reduced. In Fig. 11 we show some images of those domes in which the structure of the arch crossings is clearly appreciated.

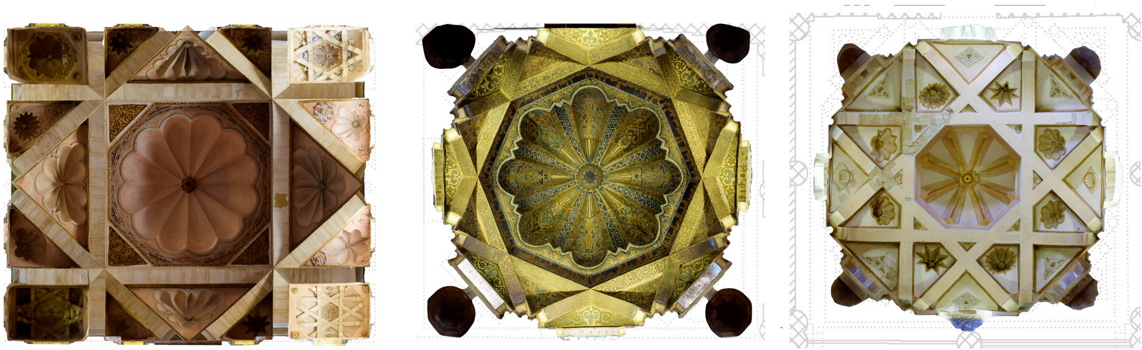


Figure 11. Photogrammetric view of the 3 types of ribbed domes. From left to right: Villaviciosa chapel, Maqsura and lateral Maqsura domes.

In Fig. 12 we show the geometrical structure of the three domes.

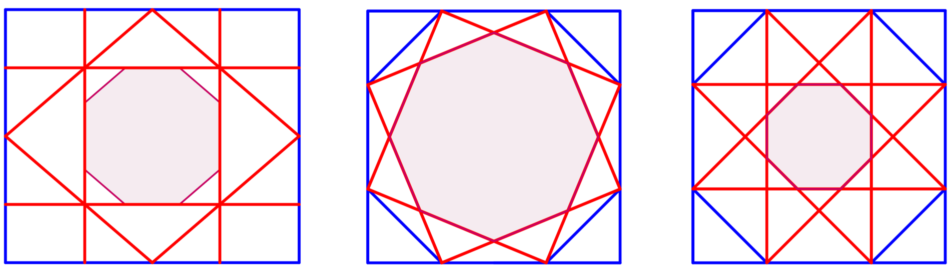


Figure 12. The geometry of the 3 ribbed domes (Villaviciosa chapel, Maqsura and lateral Maqsura domes). All with 8 arches, but with a different number of crossings (4, 8 and 12 respectively).

The different domes at the Mosque of Córdoba present different construction difficulties due to the nature of the arch crossings (Fig. 13). In the Villaviciosa chapel and the Maqsura the ribs cross each other spatially and require complex crossing pieces for their construction, in the shape of 4 or 6 quadrangular prisms joined to the faces of a cube. Those pieces accommodate the ribs that converge in them. The lateral Maqsura domes are geometrically more complicated as they have a greater number of crosspieces. However, that pieces are the easiest to replicate, since the ribs cross locally in a plane, which avoids the need for clefts or notches. This makes it possible for the stones at the intersection to fit together with simple cutouts, which are easier to produce and fit. It can be seen as an architectural version of the computational principle known as *divide and conquer* [13]. The solution to the problem of the crossings is given by dividing the problem into multiple problems of less complexity each, and then joining the solutions of all of them. This type of solution to construct a dome has been the most widely accepted in history, expanding rapidly to the rest of al-Andalus, to the Christian *mudéjar* (moorish style when Christian kings ruled) architecture, and to the European technical and cultural communication route that was the *Camino de Santiago*, in the north of the Iberian Peninsula, from where it spread to France, Great Britain, Germany, and even to the Middle East.

For these ribbed domes we have designed and printed the basic arcs that conform the domes with a series of grooves and slots that allow to mount a model of the domes in which is easy to understand the advantages and problems that each one of the structures present. In Fig. 15 we show the 3D printed and mounted pieces for each one of the 3 types of domes in the Mosque of *Córdoba*.

Figure 13. Rendering of the 3 types of ribbed domes: lattice + diamond, and stars [8/2] and [8/3].

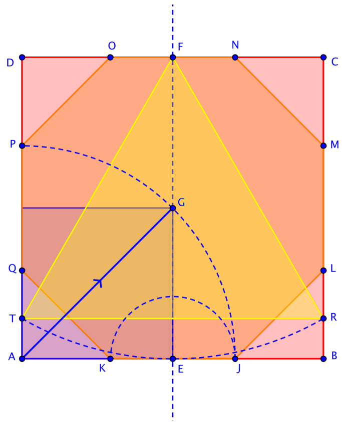
Figure 14 (left). Photograph of the Maqsura dome in the Mosque of Córdoba.
Figure 15 (right). Manufacturing with 3D printing the 3 distinct ribbed domes. Assembly of 9 pieces per dome.

4. Renaissance and baroque iconic objects

We have also designed and produced other architectural elements as 3 regular polygons inscribed one each other. These polygons are the bronze knockers on a renaissance façade at a Granada palace, *Casa de los Tiros*.

They are a triangle, a square and an octagon of precise measurements (Fig 16). The uniqueness of these three mathematical objects on the façade has to do with their symbolic meaning as basic pieces of geometry and iconic as a representation of medieval hermetic knowledge. It is very significant that other artists of the time, such as Leonardo Da Vinci or Nicoletto Rosex da Modena, implicitly or explicitly used these 3 polygons in a similar relationship with each other. In Figs. 17 and 18 we show the geometric construction of each one of the polygons and their size relations and the 3D model depiction that we have produced to easily show the proportions of each polygon.

Figure 16. 3 bronze knockers in the main façade of a renaissance palace.



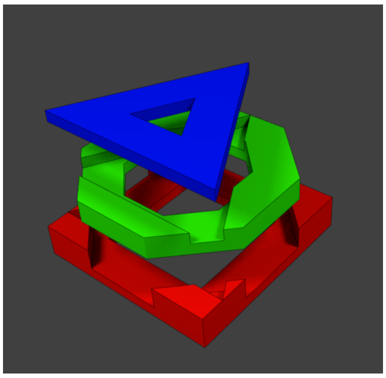
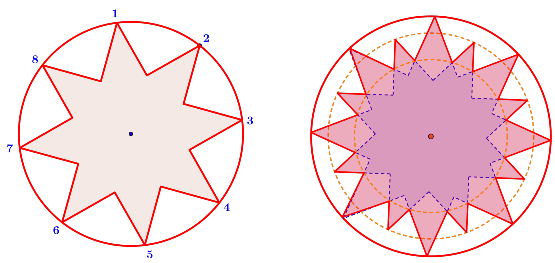


Figure 17 (left). Geometric construction of the inscribed polygons.
Figure 18 (right). Rendering of the 3D model for the polygons.

We have also designed and 3D printed the double star-shaped polygon [8/3] on Fig. 19. They have different radius for each of the stars and a twist of 22.5 degrees with respect to the other (Fig. 20). The tips (and the notches) from every star alternates between them. They can be seen as an oculus with a symbolic meaning in the baroque façade of the Cathedral of Granada. In Fig. 21 we show the produced model to show the construction of such a double star-shaped polygon from the basic star shapes.



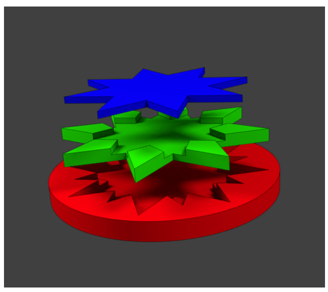


Figure 19 (left). Double star oculus in the Cathedral of Granada.
Figure 20 (middle). Construction of a star polygon [8/3] (middle left) and double rotated star polygon (middle right).
Figure 21 (right). Rendering of the double star polygons over a basis with their relative measures.

5. Laser cut mosaics

The mosaics found in walls, ceilings and floors of the Alhambra and other Nasrid (the last Islamic dynasty that ruled the kingdom of Granada) and *mudéjar* monuments, are one of the summits of 2D geometric decoration. They have motifs that tessellate the plane [1, 5, 6], exhibiting a wide variety of symmetries and rotations. So, they are ideal for explaining their geometric properties whereas solving puzzles with them.

Even more, they can be easily reproduced by laser cutting on a laminated wooden board. Once the pieces have been cut and painted with the different colors, assembling the mosaic is an excellent exercise in geometric fitting and observation of the plane isometries in the mosaic.

We have reproduced 8 different mosaics in wood. All of them are extracted from the same designs in hispanic-muslim decoration. They allow to explain and observe the concepts of translation, rotation, symmetries or glide symmetries and to materialize them in an easy way. For more advanced levels, they also allow us to talk about crystallographic groups in 2D (wallpaper groups), at least the simplest ones given by rotations and symmetries. Also some unique mosaics, containing decagonal and pentagonal stars or intertwined ribbons of Almohad origin.

In figures 22, 23 and 24 we show the wood laser cut mosaics and the corresponding designs of the Nasrid palace of the Alhambra (Granada) on which they are based. For the mathematical determination of the mosaics we follow the common notation of the International Union of Crystallography [5, 6]: *p3, p4, cmm…*

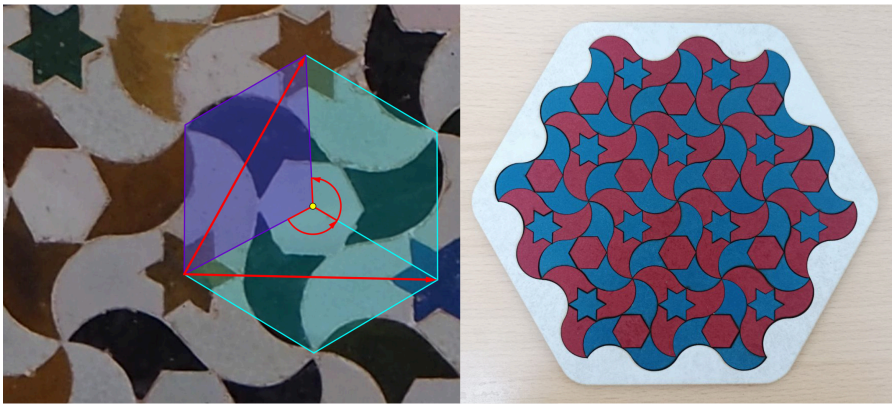


Figure 22. *The little Spanish folded paper bird mosaic*. A p3 model with only order 3 rotations (apart from translations).

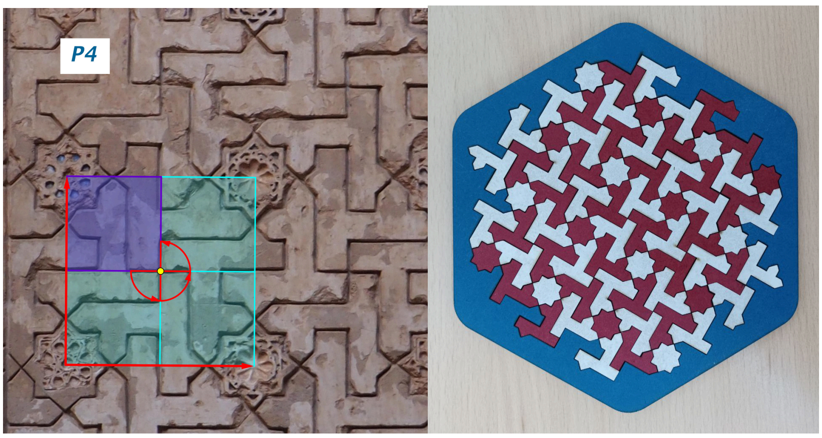


Figure 23. *The mosaic of keys*. A p4 model, with only order 4 rotations. It is an Umayyad inspiration which is often found with many possible variants.

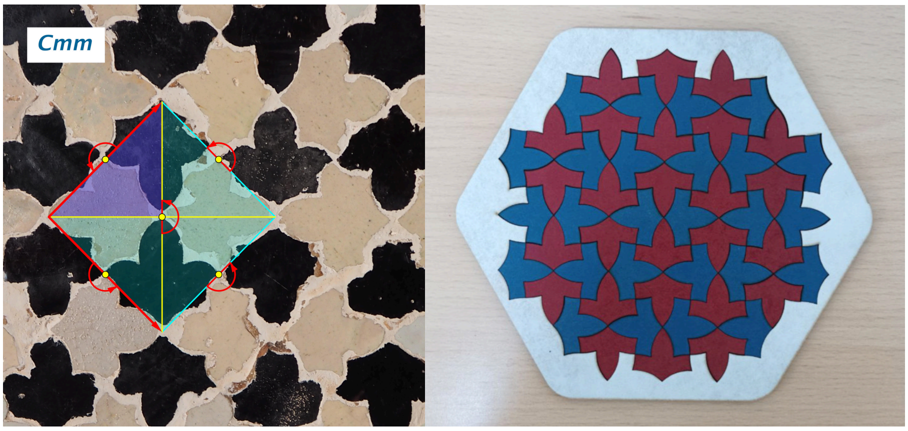


Figure 24. *The mosaic of leaves.* A cmm model, which preserves the leaves colors.

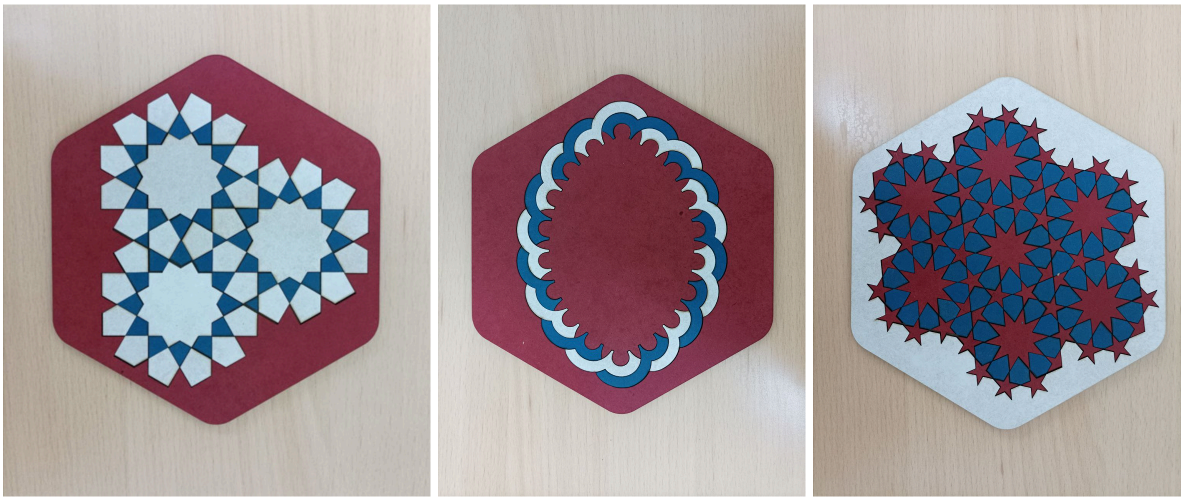


Figure 25. Other mosaics, including radial ones, Almohad straps and a quite singular tesselation with pentagonal stars.

6. Laser cut tools for direct mathematical measures

We have also made some tools in order to easily get measures of mathematical relations significant in monumental art. When we take a measure it is important for students to understand that, sometimes, we are aiming to measure not for exact quantities but for a relation among them. This is true for classical proportions that are found in many architectural monuments or the parameters that conform a horseshoe arch.

6.1. Mathematical ratios measuring kit

We have designed and laser cut a measuring kit that helps determining which ratio is used in a particular monument (door, window, façade…). We can choose from 10 different predefined proportions: 1:1 (square), 5:4, Cordoba proportion, $\sqrt{2}$, 3:2, ϕ (golden ratio), $\sqrt{3}$, 2:1 (double square), $\sqrt{5}$, and δ_S (silver ratio). Once we choose a proportion, we pick up the rectangular frame with that proportion from the kit. Then, we try to make it coincide with the architectural object we want to measure, by moving it in front of our eyes. Obviously, this will not give a very precise measurement, but can help to distinguish among the different proportions that may have been used in each monument. In Fig. 26 we show a picture of the produced and laser cut ratios measuring kit.

Figure 26. Kit of frames for direct measurement of proportions.

6.2. Tool for the measurement of horseshoe arch parameters

In the case of horseshoe arches, relative measurements of their main parameters (superelevation, thread and eccentricity) against the radius of the arch (*R*) are also important. These parameters are associated with different historical and artistic periods, as well as with the evolution among them. It is therefore important to know them from a mathematical point of view for a history of art classification. We have designed and laser cut a tool that allows to easily (and approximately) measure these parameters in relative terms (Fig. 27).

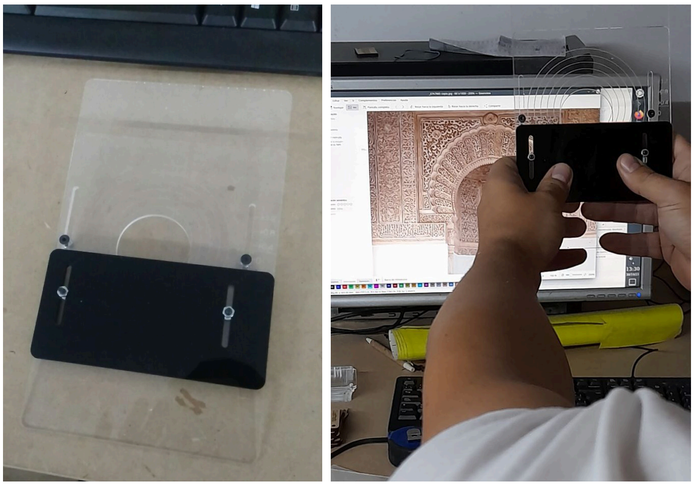


Figure 27. Tool for measurement of horseshoe arches and an example of a measure over an arch.

The presented tools show how simple measuring tools can be designed and produced easily with laser cut technology. We usually use methacrylate and wood to create different layers that are assembled with some bolts and nuts.

7. Design process for models

To design the different models we started from a previous study on the object we were dealing with. To do this, we consulted some specialized literature about the construction processes that were carried out to create the monuments and art pieces that we are going to model. Moreover, we have always tried to visit the monuments in order to get accurate measurements, photographs and even to produce our own photogrammetric models, which are of a great help to create accurate digital models, as many of the real world objects cannot not be easily accessed due to their location (ceilings, walls...) and their protection status. In fact, photogrammetry is a digital technique which is greatly increasing its application in the field of arts and restoration [7, 10].

Once the preliminary study works conclude we are ready to create the models. To do it, we have used several different digital tools that range from mathematical ones as GeoGebra [16] to 2D and 3D modeling applications as Blender [14], FreeCad [15] and Inkscape [17]. Those pieces of software are free software or at least (in the case of GeoGebra) free to use. Moreover those programs have plenty of documentation and tutorial that allows anyone to get introduced in the modeling tasks that are needed for this kind of projects.

For example, in the case of the modeling of the muqarnas vaults pieces we usually part from a simple 2D shape (half square, a rectangle…) which is extruded to form a prism. That prism is then modified using other 3D primitives as are cubes, spheres, cylinders by means of boolean operations as unions, intersections and subtractions. For example in Fig. 28 we present the modeling process of four of the pieces (*atacia*, *conza*, *almendrilla* and *medio cuadrado ciruelo*) in which from the basic prism some spheres, cylinders and cubical prisms are subtracted.

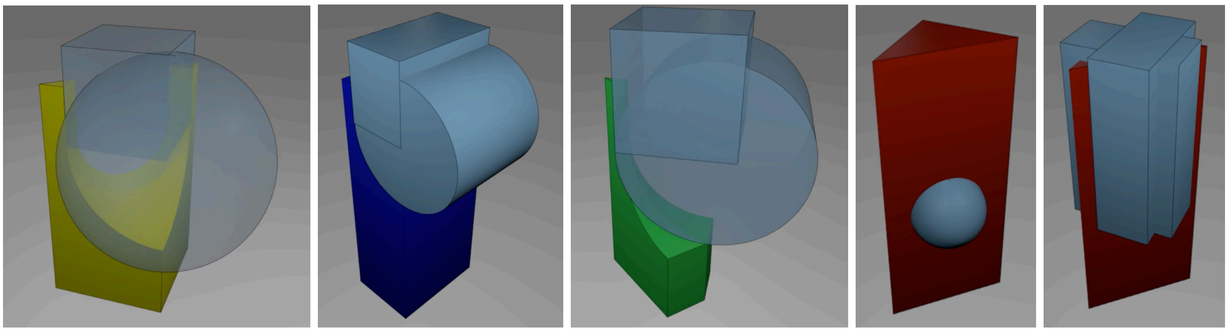


Figure 28. The modeling process for 4 muqarnas: *atacia* (yellow), *conza* (blue), *almendrilla* (green) and *medio cuadrado ciruelo* (red).

In addition to the intrinsic characteristics of each of the modeled pieces we have also included some modifications in order to help mounting the different components of each 3D printed architectural object. For example, in the case of the muqarnas ceilings we also made some holes to insert or screw some cylindrical neodymium magnets that held all pieces together (Fig. 29).

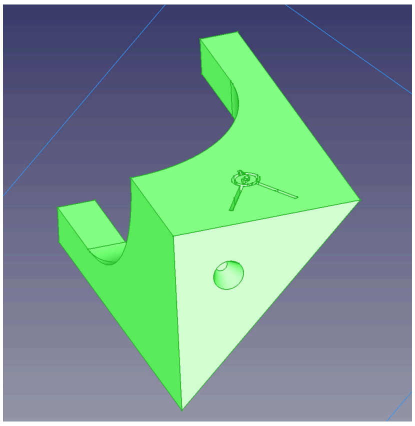


Figure 29. A hole in the bottom of the *atacia* model to screw a cylindrical neodymium magnet.

In order to attach the muqarnas together we have also designed a three level support box in which some cylindrical holes (with some metallic sheets that helps magnets to attach to) help to create the whole muqarnas structure (Fig. 30).

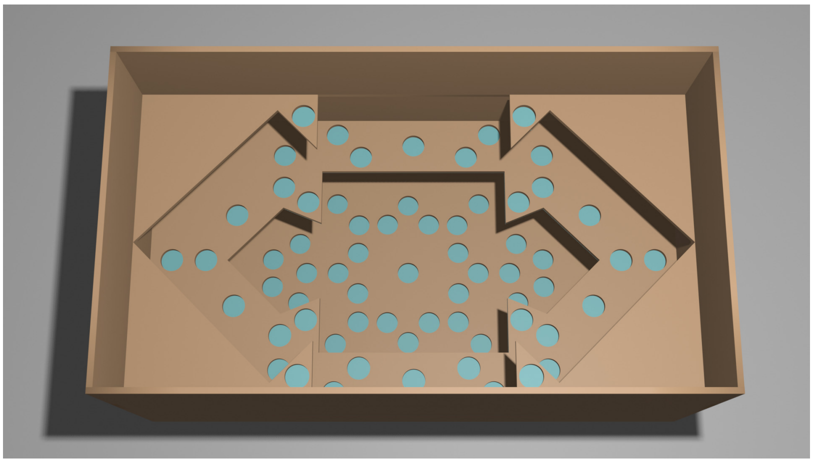


Figure 30. Support box to mount all muqarnas. The cylindrical holes allow neodymium magnets to attach to a metallic sheet (in blue).

In other models, as for example the ones in the ribbed domes, we have not only modeled the arches themselves, but we had to incorporate some cut-outs to allow to assemble the different arches together. Those cut-outs have been designed with enough tolerance to allow the full construction of the domes. In Fig. 31 we show an example of the cut-outs for the ribs of the *Villaviciosa chapel* as they are the most complex that we have produced because in each of the intersections coincide three simultaneous arches.

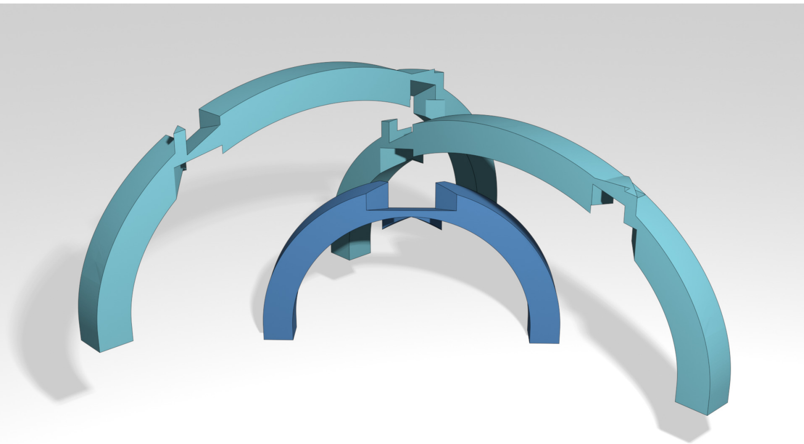


Figure 31. The modeled arches from the *Villaviciosa chapel*.

8. Fabrication technology

Once the models have been designed, we proceeded to manufacture them. We basically have used two different technologies to do it. The first one is 3D printing which in the last years have become very popular. Schools and particulars usually do have this kind of printers at hand. The most popular 3D printers to date are FDM printers. They mainly work by depositing thermoplastic that is heated (in order to melt it) and then extruded through a nozzle. The plastic is deposited as a filament forming 2D layers that one stacked together to produce the printed piece (Fig. 32).

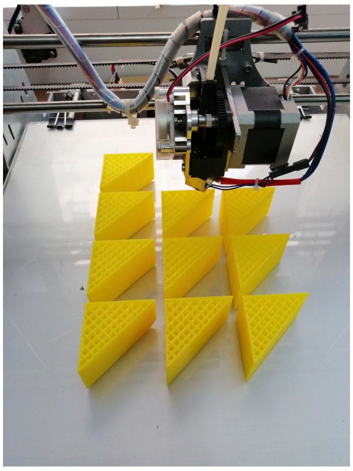
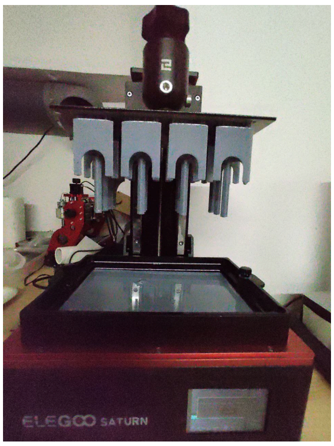


Figure 32. The FDM printer manufacturing some muqarnas models.

In recent years other 3D printing technologies have appeared and have been made accessible for the general public. For example, liquid resin printers (also called stereolithography or SLA) work also in a 2D layer by layer fashion. However, SLA printers work by curing liquid resin by means of a light source (typically in the ultra-violet wavelengths) which hardens into a solid piece of plastic (Fig. 33).

In all our pieces we have used both 3D printing technologies. We can highlight some advantages and disadvantages of each one of them:

- FDM is cheaper as the thermoplastic is usually less expensive than liquid resins.
- FDM results are not so good as SLA ones: the 2D layers are much more evident and the pieces may suffer a higher cleavage both in the printing process and when the piece is manipulated. Moreover, SLA offers higher detail which is indeed quite important when you need to ensemble several pieces together.
- SLA needs of additional steps than just the printing process: pieces have to be cleaned and cured (with ultraviolet light) in order to harden their surface, which is a tedious and dirty process (Fig. 34).
- SLA is, in general, faster than FDM: Many pieces can be printed in one single go. For example if a piece takes 1 hour to print, if we need 8 pieces with SLA we can print the 8 of them in one single 1 hour print, whilst FDM would need 8 hours of printing.



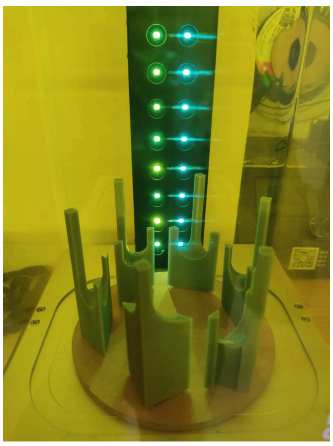


Figure 33 (left). SLA printer with a set of muqarnas models already printed.
Figure 34 (right). Curing process of the printed pieces.

For flat pieces (for example the mosaics and the measuring tools that we have developed) we have used laser cutting. Laser cutting allows precise and fast cuts of different materials. Although there exists many different kinds of laser cutting machines, allowing different materials (even thick metals) we have used less powerful ones that allow cutting different sheets of wood or MDF or even some plastic sheets like methacrylate. One of the main advantage of laser cutting is its speed, which allows to fast prototyping of the models. It also has the advantage of price, since the materials used (especially MDF) is cheap and durable. In Fig. 35 we show the process of laser cutting some mosaics.

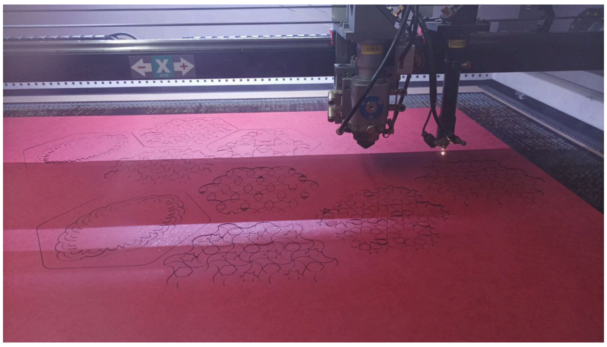


Figure 35. Laser cutting some mosaics.

Finally we must point out that even those fabrication technologies (3D printing and laser cutting) are not perfect and quite often it is necessary to do some post-processing work on the manufactured pieces. For example, in many occasions some of the pieces have to be sanded in order to remove burrs, magnets have to be glued or screwed to the pieces and they have to be painted by hand if we want them in different colors (Fig. 36).

Figure 36. Post-processing works on the pieces: sanding and attaching magnets.

9. Experiences in Mathematics Education with the produced models

With all the fabricated objects we have conducted numerous workshops and demonstrations for students. They have also been part of long training courses for math teachers, technology teachers and other specialists in art history or heritage studies. They also allow to start in the use of 3D resources through digital technologies in the classroom.

Parts are durable, reusable, and resistant to rough handling. In case of breakage, it is easy and cheap to replace them. They can be disassembled and mounted by all participants, allowing everyone to actively participate. To date, more than 100 math walks and 40 workshops have been carried out using these items (Fig. 37).

The groups that have used all these items have been very heterogeneous. For example, the laser cut mosaics have been widely used with different groups of students ranging from 8 to 16 years. With the younger ones a much more ludic approach was taken, mounting the mosaics as if they were puzzles. With older students the concepts of rotations, translations and symmetries were shown over the mosaics whilst they were mounting them.

Secondary students and adults groups were particularly interested in the most complex pieces as the ribbed domes or the muqarnas ceilings (specially in the math walks were they visited the actual monuments). Those pieces allowed the understanding of basic stellated polygons construction and diffenrent kinds of symmetries (radial, axial, and so on). Moreover, muqarnas also offered the possibility to explain to the most interested public the mathematical design process of the individual pieces by means of the application of boolean operators as intersection or substraction of simple 3D objects (cylinders, prisms and so on).

The groups sizes ranged between 15 and 30 attendees. When groups were small in size it has been also feasible to introduce the tools for measuring proportions. They easily allow to understand what a particular ratio means in an artistic object and how those ratios can be found in almost every monument.

Figure 37. Examples showing the use of 3D printed objects in math walks and demonstrations.

The main mathematical topics that are covered in these experiences using the fabricated objects change according to the level of the students. For example, we have talked about 2D and 3D symmetries, reflections, translations, movement groups of a plane, proportions, concavity and convexity, regular and stellated polygons, spaces coverage by prisms, arches and their parameters, arch crossings, curves in 3D space and spatial lattices.

In addition, the manipulation of the manufactured parts allows a clear appreciation of the possibilities of the application of mathematics in fields such as architecture or engineering. Moreover, students have shown a greater interest in these digital technologies.

From those workshops and math walks we have deducted some facts:

- In the workshops we have challenged the students to assemble the 3D objects like a puzzle. This challenge have increased their focus and attention to the mathematical aspects of the pieces (Fig. 38).
- The assessment of the carried out courses and activities has been very satisfactory in all cases.
- Elementary or middle school students never turn down a puzzle challenge. Students demonstrate that they understand the mathematical knowledge involved in the challenge.

- Cooperation between students, which spontaneusly tend to form small groups, arises naturally when taking on these challenges. What one student “sees” (deduces and understands) is shared. Therefore, a collaborative team approach is easily established.
- In the **mathematical walks**, the simple act of showing the models has greatly increased the interest of the participants, breaking at some point the barrier that usually exist among the attendants and the guide. Usually the feedback about using these materials during the walks and explanations has been very positive.
- In the **long courses** for teachers and specialists, they have valued the activities with 3D printing and laser cutting and have stated them as a very attractive and transferable alternative to their own projects with their students.

Figure 38. Examples of activities with students and the laser cut and 3D models.

With those experiences we can conclude that objects that can be manipulated are always exciting and allow to directly challenge the spatial vision of the students and their geometric understanding of the objects. The students on those workshops are usually in the 12-16 age range, but we have also satisfactorily tested this experiences with younger students.

10. Conclusions

In this paper we have shown how several architectural objects can be interesting to be modelled and 3D printed in order to obtain objects that can be used to teach the mathematical properties applied in art and monuments. Those models minimize the inherent difficulty of the visualization of the interrelation of their parts. Particularly we have shown the modeling and manufacturing of a section of a muqarnas vault and ribbed domes, typical of Islamic architecture. We have also dealt with objects of different artistic periods as the Renaissance and Baroque ones. Specially the latter, due to the mathematical nature of their artistic works, are very suitable for 3D modeling and printing.

The modeling process involves, apart from the mathematical study, a deep understanding of the spatial construction that includes extrusion, aggregation of surfaces and volumes, superimposition of parts and so on. Moreover, for those models being usable it is also necessary to pay attention to some engineering steps involving cut-outs, addition of magnets and supports, etc.

The mathematical topics that have been addressed are about 2D and 3D symmetries, reflections, translations, movement groups of a plane, proportions, concavity and convexity, regular and stellated polygons, spaces coverage by prisms, arches and their drawings parameters such as eccentricity or superelevation, arch crossings, curves in 3D space, spatial lattices and boolean operations with surfaces.

Some of the advantages offered by the use of 2D and 3D parts are the attractiveness of their manipulation, which leads to a closer personal involvement, as well as the possibility of experimentation. They help to include in the mathematical curriculum topics of geometry, curves or arithmetic calculation with a visual and instrumental approach, more direct and easy to understand and handle. They are also well suited to provide an understanding of 3D spatial arrangements that are otherwise difficult to explain. The relationship between the mathematical object and the manufactured part is established as a very useful pairing, and brings mathematics closer to the field of prototyping and experimentation.

Our kits are mathematical products with links to the world of monumental art, engineering and technology, making them a true STEAM product. Their ability to arouse interest and contextualize in an interdisciplinary framework will help the mathematical understanding and make these technologies a companion in the evolution of mathematics education.

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Software Programs

[14] Blender, a free 3D creation suite. https://www.blender.org
[15] FreeCad, your own 3D parametric modeler. https://www.freecadweb.org/
[16] GeoGebra*, A dynamic mathematics software*. https://www.geogebra.org
[17] Inkscape, a free 2D free design tool. https://inkscape.org/es/